Also solved by Robert Agnew, Hongwei Chen, Marty Getz & Yuanyuan Zhao, Tom Jager, Moubinool Omarjee (France), Paolo Perfetti (Italy), San Francisco University HS Problem Solving Group, Mehtaab Sawhney, Nicholas C. Singer, and the proposer. There were 3 incomplete or incorrect solutions.

An inequality between symmetric polynomials **June 2015**

1973. *Proposed by Arkady Alt, San Jose, California, USA.*

Let $\Delta(x, y, z) = 2(xy + yz + xz) - (x^2 + y^2 + z^2)$. Prove that for any positive real numbers a, b, and c, the following inequality holds:

$$\left(\Delta(a^2, b^2, c^2)\right)^2 \geq \Delta(a, b, c) \cdot \Delta(a^3, b^3, c^3).$$

Solution by Mehtaab Sawhney (student), Commack High School, Commack, NY.
We must prove $D(a, b, c) \geq 0$, where

$$D(a, b, c) = \left(\Delta(a^2, b^2, c^2)\right)^2 - \Delta(a, b, c) \cdot \Delta(a^3, b^3, c^3).$$

Since D is symmetric in the variables a, b, c, we may assume $a \leq b \leq c$ without loss of generality. Let $u = b - a$ and $v = c - a$; then we have $0 \leq u \leq v$. Let $P(a, u, v) = D(a, a + u, a + v)$, and let $\delta = v - u$. Direct computation shows that P is a polynomial in a, u, v of degree 6 in a, namely

$$P(a, u, v) = \sum_{j=0}^{6} Q_j(u, v)a^j.$$

The coefficients $Q_j = Q_j(u, v)$ are themselves polynomials in u, v, given by

$$Q_0 = \delta^4 uv\left(2u^2 + 3uv + 2v^2\right), \qquad Q_3 = 20\delta^2(u + v)(2u^2 + uv + 2v^2),$$

$$Q_1 = 2\delta^2(2u^5 + 7u^4v + 7uv^4 + 2v^5), \qquad Q_4 = 10\delta^2(4u^2 + 5uv + 4v^2),$$

$$Q_2 = 20\delta^2(u^4 + 2u^3v + 2uv^3 + v^4) + 19\delta^2u^2v^2, \qquad Q_5 = 2(u + v)(10\delta^2 + uv),$$

$$Q_6 = 4(\delta^2 + uv).$$

Since $u \geq v \geq 0$, we also have $\delta = v - u \geq 0$, hence $Q_j(u, v) \geq 0$, and so $D(a, b, c) = P(a, b - a, c - a) \geq 0$ for $0 \leq a \leq b \leq c$ as we sought to show.

Editor's Note. The inequality holds provided $a, b, c \geq 0$, as follows from the proof above, or else by continuity from the case $a, b, c > 0$.

Also solved by Andrea Fanchini (Italy), Kee-Wai Lau (Hong Kong), Paulo Perfetti (Italy), Nicholas Singer, and the proposer. There was 1 incomplete or incorrect solution.

A sum of reciprocals of q-polynomials **June 2015**

1974. *Proposed by Boon Wee Ong, Behrend College, Erie, PA.*

Let $q \neq 1$ be a positive real number. Define for $n \geq 1$,

$$\nu_n = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \quad \text{and} \quad \mu_n = q^{n/2} + q^{-n/2}.$$